The Commutative Quaternion Number System

CaydendW

2 June 2022

Contents

Abstract

The quaternions are a number system that have 1 real part and 3 imaginary parts. They were invented by Sir William Rowan Hamilton and are used majorly in computing to describe 3rd and 4th dimensional angles. As you move from the friendly complex numbers to the quaternions, you lose commutativity. So, this system has quaternions that are completely commutative. They lose a lot of nice things quaternions give and I really doubt they are useful in the slightest. This is just a simple little passion project that I hope you enjoy laughing at. This paper just sets out the basic rules representations of the number system and a few common problems in maths that are solved with these numbers.

Notes

This number deals with commutative quaternions: my brand of quaternions. Whenever the imaginary units i, j or k are mentioned, assume that they are the commutative quaternion i, j and k unless otherwise stated. Any number being passed into a function having to do with these numbers should also be assumed to be part of the commutative quaternions. Any number expressed as a coefficient of one of the commutative quaternion units (i, j and k) can be assumed to be real unless otherwise stated.

In this paper, the distance function/modulus (Denoted by $||q||$) and the norm/isotropic quadratic form (Denoted by $N(q)$) are ***not*** used interchangeably. A distinction is made between the 2 functions and they should not be confused with one another.

Chapter 1

Definitions and basic proofs

1.1 Basic definition

Commutative quaternions are a commutative and associative algebra consisting of 1 real part and 3 imaginary parts.

A commutative quaternion expressed as an expression in the form:

$$a + bi + cj + dk$$

Where a, b, c and d are real numbers and i, j and k are the imaginary parts of the commutative quaternions that satisfy the following rules:

$$i^2 = -j^2 = k^2 = -ijk = -1$$

And by association:

$$\begin{aligned} ik &= ki = j \\ ji &= ij = -k \\ jk &= kj = -i \end{aligned}$$

In modern mathematical language, the commutative quaternions form a 4 dimensional commutative, associative, composition, split algebra over the real numbers but it does not form a domain and is not a division algebra (Due to it having non-trivial 0 divisors).
The commutative quaternions are closed over powers, roots and logarithms and are

almost a field but they aren't exactly because they contain non-trivial 0 divisors, thus making them not a field.
This algebra has non-trivial 0 divisors, therefore it is a "split" algebra and should technically be called the "split commutative quaternions" but this is a bit long of a name so the "split" part will be ignored.

The set of all commutative quaternions (Expressed as the symbol $\mathbb{X}$) can be expressed as:

$$\mathbb{X} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R};\ i^2 = -j^2 = k^2 = -ijk = -1\}$$

It is worth noting that:

$$\mathbb{R} \subset \mathbb{X}$$

Formula 1 (Square of a commutative quaternion)**.** *The square of a commutative quaternion looks like:*

$$\begin{aligned}
q &:= a + bi + cj + dk \\
q^2 &= (a + bi + cj + dk)(a + bi + cj + dk) \\
q^2 &= a^2 + abi + acj + adk \\
&\quad + abi + b^2i^2 + bcij + bdik \\
&\quad + acj + bcij + c^2j^2 + cdjk \\
&\quad + adk + bdik + cdjk + d^2k^2 \\
q^2 &= a^2 - b^2 + c^2 - d^2 + 2abi + 2acj + 2adk + 2bdj - 2bck - 2cdi \\
q^2 &= a^2 - b^2 + c^2 - d^2 + (2ab - 2cd)i + (2ac + 2bd)j + (2ad - 2bc)k
\end{aligned}$$

Formula 2 (Square of a vector commutative quaternion)**.** *The square of a vector commutative quaternion looks like:*

$$\begin{aligned}
q &:= bi + cj + dk \\
q^2 &= (bi + cj + dk)(bi + cj + dk) \\
q^2 &= -b^2 - bck + bdj - bck + c^2 - cdi + bdj - cdi - d^2 \\
q^2 &= -b^2 + c^2 - d^2 - 2cdi + 2bdj - 2bck
\end{aligned}$$

1.2 Matrix representations

Commutative quaternions can also be represented as matrices just like regular quaternions and complex numbers:

$$a+bi+cj+dk \equiv a\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}+b\begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}+c\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}$$
$$+d\begin{bmatrix}0&0&1&0\\0&0&0&-1\\-1&0&0&0\\0&1&0&0\end{bmatrix}$$
$$a+bi+cj+dk \equiv \begin{bmatrix}a&-b&d&c\\b&a&c&-d\\-d&c&a&b\\c&d&-b&a\end{bmatrix}$$

There are more than 1 of these matrix representations for this number system as will be demonstrated.

Every 4×4 corresponds to a multiplication table of the commutative quaternions. For example the table that corresponds to the above listed matrix:

$\times$	a	$-b$	d	c
a	a	$-b$	d	c
b	b	a	c	$-d$
$-d$	$-d$	c	a	b
c	c	d	$-b$	a

Is isomorphic (Through $\{a\mapsto 1,\ b\mapsto i,\ c\mapsto j, d\mapsto k\}$) to:

$\times$	1	$-i$	j	k
1	1	$-i$	j	k
i	i	1	$-k$	j
j	j	k	1	$-i$
$-k$	$-k$	j	i	1

Therefore, any table that satisfies the multiplication rules and represents the number 1 as the identity can be a matrix representation.

Note how the matrix for j is similar to that of the split complex number j. It also shares the same property of when being squared it equals to one. Both of them are just anti-diagonal matrices

Also note how the matrices for i and k are similar. They are just reflections of one another and I believe that if you switch i out for k, no generality will be lost.

As a curiosity, all the matrices are also orthogonal (Unproven).

Another good matrix representation (Although somewhat limited in some ways) is by having a 2×2 matrix with complex parts to represent the numbers. A representation would look something like this (Note: The i in the matrices is the complex number i and not the commutative quaternion i. They are distinct):

$$
\begin{aligned}
a + bi + cj + dk &\equiv a\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + b\begin{bmatrix}i & 0\\0 & i\end{bmatrix} + c\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} + d\begin{bmatrix}0 & i\\i & 0\end{bmatrix}\\
a + bi + cj + dk &\equiv \begin{bmatrix}a+bi & -c+di\\-c+di & a+bi\end{bmatrix}
\end{aligned}
$$

1.3 Proof of commutativity

Claim. The commutative quaternions are commutative.

Proof. To prove that the commutative quaternions are commutative, you just have to prove that the matrices commute. Multiply them together and observe that they yield the same result regardless of the order

$$ik = j = ki$$

$$\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$ji = ij = -k$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$jk = kj = -i$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Q.E.D.

And of course, the identity matrix, or number 1, is always commutative (It is a scalar matrix) and does not have to be proven.

You could also just prove this using simultaneous diagonalisation but this is easier.

Chapter 2

Operations, functions and properties

2.1 Custom function definitions

These are functions that are somewhat custom yet easy to define.

$$
\begin{aligned}
q &:= a + bi + cj + dk \\
\mathrm{Re}(q) &:= \mathrm{Sc}(q) := a \\
\mathrm{Im}(q) &:= \mathrm{Ve}(q) := bi + cj + dk \\
\mathrm{Im}_i(q) &:= \mathrm{Ve}_i(q) := b \\
\mathrm{Im}_j(q) &:= \mathrm{Ve}_j(q) := c \\
\mathrm{Im}_k(q) &:= \mathrm{Ve}_k(q) := d
\end{aligned}
$$

As can be seen, the standard Re and Im functions are used, as well as functions reminiscent of Hamilton's functions: Ve (vector part) and Sc (scalar part).

2.2 Isotropic quadratic form or norm ($N(q)$)

Note: In this paper, the distance function/modulus (Denoted by $||q||$) and the norm/isotropic quadratic form (Denoted by $N(q)$) are ***not*** used interchangeably. A distinction is made between the 2 functions and they should not be confused with one another. Their relationship and interaction is defined later.

The norm function will be defined as the N function. And from here on out, $N(q)$ is the norm of q.

Formula 3 (Definiton of the norm). *The N function shall be defined as:*

$$N(q) := q\overline{q}$$

Lemma 1 (Conditions of the norm). *Due to the fact that the commutative quaternions are a composition algebra, the following properties must be satisfied:*

$$\forall x, \exists \overline{x} : x\overline{x} = N(x)$$
$$N(qz) = N(q)N(z)$$

This definition is useful for later as right now, we don't have a definition for the conjugate of a number.

A little bit of research on the N function shows that the N function is just the determinant the matrix representation of the number.[1] So, we can say:

$$\begin{aligned}
&\det\left(\begin{bmatrix} a & -b & d & c \\ b & a & c & -d \\ -d & c & a & b \\ c & d & -b & a \end{bmatrix}\right) \\
=&((a+c)^2 + (b-d)^2)((a-c)^2 + (b+d)^2) \\
=&(a - b^2 - c^2 + d^2)^2 + (2ab + 2cd)^2 \\
=&a^4 + b^4 + c^4 + c^4 + 2a^2b^2 - 2a^2c^2 + 2a^2d^2 + 2b^2c^2 - 2b^2d^2 + 2c^2d^2 + 8abcd
\end{aligned}$$

Which is to say, not as pretty as the regular quaternion's N function but surprisingly neat.

Issue is, if we take the standard definition of the distance function:

$$||q||^2 = N(q)$$

We notice that a known result like the distance of $1 + i$ is wrong (It should be $\sqrt{2}$ but our N function yields 2). This is due to the fact that we're using 4×4 matrices and the effect is the same as the quaternions' 4×4 matrices: the determinant of the 4×4 representation of a number, is actually the N function squared. Therefore, the N function should be the square root of the determinant as to yield the correct result.

Formula 4 (Algebraic definition of the norm). *The norm shall be defined as:*

$$q := a + bi + cj + dk$$
$$N(q) := \sqrt{((a+c)^2 + (b-d)^2)((a-c)^2 + (b+d)^2)} = q\bar{q}$$

We expect negative numbers from this function when $d > b, c > a$.

2.3 Modulus or distance function ($||q||$)

The definition of the distance function in these complex number systems is pretty much always:

$$||q||^2 := N(q) = q\bar{q}$$

But this runs into the issue of taking square roots of negative numbers because the N function can give back negative numbers. This problem is also a problem of the split complex numbers. Depending on what source you consult, you either see them define the distance function differently or saying that the distance function is not positive defined and having a signature $(1, -1)$ and having it not be a true mathematical norm. It makes sense to use the first option so that you can later define a working planar form. Thus, the definition of the modulus shall be the same.

Formula 5 (Definiton of the modulus). *The modulus shall be defined as:*

$$||q||^2 := N(q) = q\bar{q}$$

Since we have a distance function that can return imaginary or negative results, this is not a true mathematical norm.

This definition is sound because it fits in with the split complex numbers. You can have a distance of -1 away from the origin. It is just the conjugate hyperbola and it is actually the only way to represent the split complex number j.

Finding the distance function from here is trivial.

Formula 6 (Algebraic definiton of the modulus)**.** *In algebraic terms, the norm is defined as:*

$$\begin{aligned}||q|| &= \sqrt{N(q)}\\ &= \sqrt{\sqrt{((a+c)^2+(b-d)^2)((a-c)^2+(b+d)^2)}}\\ &= \sqrt[4]{((a+c)^2+(b-d)^2)((a-c)^2+(b+d)^2)}\end{aligned}$$

Claim. The distributivity over multiplication property from the N function is preserved here.

Proof.

$$\begin{aligned}q, z &\in \mathbb{X}\\ N(qz) &= N(q)N(z)\\ ||qz||^2 &= ||q||^2 \cdot ||z||^2\\ \sqrt{||qz||^2} &= \sqrt{||q||^2 \cdot ||z||^2}\\ ||qz|| &= ||q|| \cdot ||z||\end{aligned}$$

Q.E.D.

This distance function is strange due to it not being a true mathematical norm, it should be able to return negative values and "complex" (Numbers with a square root of -1) values as well.

2.4 Conjugation ($\overline{q}$)

Using our definition of the N function, we can derive a formula for the conjugate of a number:

$$\begin{aligned}q &:= a + bi + cj + dk\\ N(q) &= q\overline{q}\\ \overline{q} &= \frac{N(q)}{q}\end{aligned}$$

As is, it's actually to difficult to solve for. Even a computer isn't able to solve that (Due to the surd in N's definition). So, instead let's define a stepping stone function $L(q)$ to make the process easier.

Formula 7 (Definiton of the L function)**.** *The L function shall be:*

$$L(q)q := N(q)^2$$

This definition allows us to solve for something without surds, making it easier to solve for with a computer.

So let's solve for L. This is done best by using the matrix representations of the commutative quaternions (Keep in mind that $N(q)^2$ is the determinant of the matrix representation of q):

$$
\begin{aligned}
q &:= a + bi + cj + dk \\
q \equiv M_q &:= \begin{bmatrix} a & -b & d & c \\ b & a & c & -d \\ -d & c & a & b \\ c & d & -b & a \end{bmatrix} \\
N(q)^2 \equiv M_r &:= \begin{bmatrix} \det(M_q) & 0 & 0 & 0 \\ 0 & \det(M_q) & 0 & 0 \\ 0 & 0 & \det(M_q) & 0 \\ 0 & 0 & 0 & \det(M_q) \end{bmatrix} \\
L(q) \equiv M_l &:= \begin{bmatrix} w & -x & z & y \\ x & w & y & -z \\ -z & y & w & x \\ y & z & -x & w \end{bmatrix} \\
qL(q) &= N(q)^2 \\
\therefore\ M_q \cdot M_l &= M_r \\
\delta_1 &:= aw - bx + cy - dz \\
\delta_2 &:= ax + bw - cz - dy \\
\delta_3 &:= ay + bz + cw + dx \\
\delta_4 &:= az - by - cx + dw \\
M_l \cdot M_q &= \begin{bmatrix} \delta_1 & -\delta_2 & \delta_4 & \delta_3 \\ \delta_2 & \delta_1 & \delta_3 & -\delta_4 \\ -\delta_4 & \delta_3 & \delta_1 & \delta_2 \\ \delta_3 & \delta_4 & -\delta_2 & \delta_1 \end{bmatrix} \\
&\therefore aw - bx + cy - dz = \det(M_q) = N(q)^2 = ((a+c)^2 + (b-d)^2)((a-c)^2 + (b+d)^2) \\
&\quad ax + bw - cz - dy = 0 \\
&\quad ay + bz + cw + dx = 0 \\
&\quad az - by - cx + dw = 0
\end{aligned}
$$

Now, we have a system of equations we can use to solve for w, x, y and z: the members of L function. Due to how the equations are laid out, this is actually solved easiest using Cramer's rule using a computer. And the final results are:

$$\begin{aligned}
w =&(a(a^2+b^2-c^2+d^2)+2bcd)\\
&a^3-ac^2+ab^2+ad^2+2bcd\\
x =&-((b(a^2+b^2+c^2-d^2)+2acd)\\
&-b^3-a^2b-bc^2+bd^2-2acd\\
y =&(c(-a^2+b^2+c^2+d^2)+2abd)\\
&c^3-a^2c+b^2c+cd^2+2abd\\
z =&-(d(+a^2-b^2+c^2+d^2)+2abc)\\
&-d^3-a^2d+b^2d-c^2d-2abc
\end{aligned}$$

Formula 8 (Algebraic definition of the L function)**.**

$$\begin{aligned}
L(a+bi+cj+dk) :=& (a(a^2+b^2-c^2+d^2)+2bcd)\\
&-(b(a^2+b^2+c^2-d^2)+2acd)i\\
&+(c(-a^2+b^2+c^2+d^2)+2abd)j\\
&-(d(a^2-b^2+c^2+d^2)+2abc)k
\end{aligned}$$

Claim. The L function of a commutative quaternion is equivilant to the cofactor matrix of the real transpose of the matrix representation of that same commutative quaternion. That is $M \equiv q, L(q) \equiv \text{cofactor}(M^T)$

Proof.

$$
\begin{aligned}
q &:= a + bi + cj + dk \\
L(q) &= (a(a^2 + b^2 - c^2 + d^2) + 2bcd) \\
&\quad - (b(a^2 + b^2 + c^2 - d^2) + 2acd)i \\
&\quad + (c(-a^2 + b^2 + c^2 + d^2) + 2abd)j \\
&\quad - (d(a^2 - b^2 + c^2 + d^2) + 2abc)k \\
q \equiv M_q &:= \begin{bmatrix} a & -b & d & c \\ b & a & c & -d \\ -d & c & a & b \\ c & d & -b & a \end{bmatrix} \\
\delta_1 &:= a^3 + ab^2 - ac^2 + ad^2 + 2bcd \\
\delta_2 &:= -a^2b - 2acd - b^3 - bc^2 + bd^2 \\
\delta_3 &:= -a^2c + 2abd + b^2c + c^3 + cd^2 \\
\delta_4 &:= -a^2d - 2abc + b^2d - c^2d - d^3 \\
\operatorname{cofactor}\left(M_q^T\right) &= \begin{bmatrix} \delta_1 & -\delta_2 & \delta_4 & \delta_3 \\ \delta_2 & \delta_1 & \delta_3 & -\delta_4 \\ -\delta_4 & \delta_3 & \delta_1 & \delta_2 \\ \delta_3 & \delta_4 & -\delta_2 & \delta_1 \end{bmatrix} \\
\operatorname{Re}(L(q)) &= \delta_1 \\
\operatorname{Ve}_i(L(q)) &= \delta_2 \\
\operatorname{Ve}_j(L(q)) &= \delta_3 \\
\operatorname{Ve}_k(L(q)) &= \delta_4 \\
\therefore L(q) &\equiv \operatorname{cofactor}\left(M_q^T\right)
\end{aligned}
$$

Q.E.D.

Now, solving for the conjugate is rather easy:

$$\begin{aligned} q &:= a + bi + cj + dk \\ qL(q) &= N(q)^2 \\ qL(q) &= (q\overline{q})^2 \\ qL(q) &= q^2\overline{q}^2 \\ L(q) &= q\overline{q}^2 \\ L(q) &= N(q)\overline{q} \\ \overline{q} &= \frac{L(q)}{N(q)} \end{aligned}$$

And that's the definition of the conjugate. The problem is, numbers with an N of 0 (Or, 0 divisors) don't have a conjugate via this definition. Since these 0 divisors should have a conjugate that together multiply to 0 as per the conditions laid out in Lemma 1, we need to define conjugates for these 0 divisors. This is challenging and the answer was found through trial and error (Proof given later). In the end, the definition of the conjugate was found.

Formula 9. *The definition of the conjugate of any arbitrary commutative quaternion shall be:*

$$q := a + bi + cj + dk$$
$$\overline{q} = \begin{cases} a - bi - cj + dk & N(q) = 0 \\ \frac{L(q)}{N(q)} & N(q) \neq 0 \end{cases}$$

A thing to notice about the conjugation formula of 0 divisors is it's actually just inverting the signs of the middle 2 terms.

2.5 Properties of the norm and conjugates

2.5.1 N of a number's L function

This is a useful thing to have as it helps with later proofs. It can be very painful to work out but it is needed.

Claim. $N(L(q)) = N(q)^3$

Proof.

$$
\begin{aligned}
q :=& a + bi + cj + dk \\
q \equiv& M_q := \begin{bmatrix} a & -b & d & c \\ b & a & c & -d \\ -d & c & a & b \\ c & d & -b & a \end{bmatrix} \\
L(q) \equiv& \operatorname{cofactor}(M_q^T) \\
N(L(q)) =& \sqrt{\det(\operatorname{cofactor}(M_q^T))} \\
N(L(q))^2 =& \det(\operatorname{cofactor}(M_q^T)) \\
N(L(q))^2 =& (a^4 + 2a^2(b^2 - c^2 + d^2) + 8abcd + b^4 + 2b^2(c^2 - d^2) + (c^2 + d^2)^2) \\
& (a^8 + 4a^6(b^2 - c^2 + d^2) + 2a^4(3b^4 - 2b^2(c^2 - d^2) + 3c^4 - 2c^2d^2 + 3d^4) \\
& + 4a^2(b^6 + b^4(c^2 - d^2) + b^2(-c^4 + 22c^2d^2 - d^4) - c^6 - c^4d^2 + c^2d^4 + d^6) \\
& + bcd(16a^5 + 32a^3(b^2 - c^2 + d^2) + 16a(b^4 + 2b^2(c^2 - d^2) + c^4 + 2c^2d^2 + d^4)) \\
& + b^8 + 4b^6(c^2 - d^2) + 2b^4(3c^4 - 2c^2d^2 + 3d^4) + 4b^2(c^6 + c^4d^2 - c^2d^4 - d^6) \\
& + c^8 + 4c^6d^2 + 6c^4d^4 + 4c^2d^6 + d^8) \\
N(L(q))^2 =& (((a + c)^2 + (b - d)^2)((a - c)^2 + (b + d)^2))^3 \\
N(L(q))^2 =& N(q)^6 \\
N(L(q)) =& N(q)^3
\end{aligned}
$$

Q.E.D.

2.5.2 Applying the L function repeatedly

This is more of a curiosity than anything useful but it's interesting nonetheless.

Claim. $L(L(q)) = qN(q)^4 = q^3L(q)^2$

Proof.

$$\begin{aligned}
q &:= a + bi + cj + dk \\
L(q)q &= N(q)^2 \\
L(q) &= \frac{N(q)^2}{q} \\
L(L(q)) &= \frac{N(L(q))^2}{L(q)} \\
L(L(q)) &= \frac{N(q)^6}{L(q)} \\
L(L(q)) &= \frac{N(q)^6}{\frac{N(q)^2}{q}} \\
L(L(q)) &= \frac{qN(q)^6}{N(q)^2} \\
L(L(q)) &= qN(q)^4 \\
L(L(q)) &= q(qL(q))^2 \\
L(L(q)) &= q^3L(q)^2
\end{aligned}$$

Q.E.D.

2.5.3 N of a real number

This is a simple result which has to proven for the following proof.

Claim. The norm of a real number in the commutative quaternions is the same as in the real numbers. That is, $N(a) = a^2$, a being a scalar commutative quaternion (Equivalent to a real number).

Proof.

$$\begin{aligned}
q &= a + 0i + 0j + 0k \\
N(q) &= \sqrt{((a-0)^2 + (0+0)^2)((a+0)^2 + (0-0)^2)} \\
N(q) &= \sqrt{a^4} \\
N(q) &= a^2
\end{aligned}$$

Q.E.D.

Which matches the real number's N function. Further proving that $\mathbb{R} \subset \mathbb{X}$.

2.5.4 N of a number's conjugate

Like most complex number systems, the N of a number's conjugate is equal to the N of the number. This is easy to prove using previous proofs.

Claim. $N(q) = N(\overline{q})$

Proof. For non 0 divisors:

$$
\begin{aligned}
q &:= a + bi + cj + dk \\
\overline{q} &= \frac{L(q)}{N(q)} \\
N(\overline{q}) &= N\left(\frac{L(q)}{N(q)}\right) \\
N(\overline{q}) &= N(L(q)N(q)^{-1}) \\
N(\overline{q}) &= N(L(q))N(N(q)^{-1})) \\
N(\overline{q}) &= N(q)^3 N(N(q)^{-1})) \\
N(\overline{q}) &= N(q)^3 N(q)^{-2} \\
N(\overline{q}) &= N(q)
\end{aligned}
$$

For 0 divisors:

$$
\begin{aligned}
&\text{Let } q \text{ be a 0 divisor} \\
&N(q) = 0 \\
&N(\overline{q}) = 0
\end{aligned}
$$

This relies on Lemma 4 to prove the above result. Q.E.D.

This is a very useful result to have for the next part.

2.5.5 Repeated conjugation

Claim. Conjugation is an involution. That is, $q = \overline{\overline{q}}$.

Proof. For non 0 divisors:

$$
\begin{aligned}
q\overline{q} &= N(q) \\
\overline{q} &= \frac{N(q)}{q} \\
\overline{\overline{q}} &= \frac{N(\overline{q})}{\overline{q}} \\
\overline{\overline{q}} &= \frac{N(\overline{q})}{\frac{N(q)}{q}} \\
\overline{\overline{q}} &= \frac{qN(\overline{q})}{N(q)} \\
\overline{\overline{q}} &= \frac{qN(q)}{N(q)} \\
\overline{\overline{q}} &= q
\end{aligned}
$$

For 0 divisors:

$$
\begin{aligned}
q &:= a + bi - aj + dk \\
\overline{q} &= a - bi + aj + dk \\
\overline{\overline{q}} &= a + bi - aj + dk
\end{aligned}
$$

$$
\begin{aligned}
q &:= a - bi + aj + dk \\
\overline{q} &= a + bi - aj + dk \\
\overline{\overline{q}} &= a - bi + aj + dk
\end{aligned}
$$

$$
\begin{aligned}
q &:= -a + bi + aj + dk \\
\overline{q} &= -a - bi - aj + dk \\
\overline{\overline{q}} &= -a + bi + cj + dk
\end{aligned}
$$

$$
\begin{aligned}
q &:= a + bi + aj - dk \\
\overline{q} &= a - bi - aj - dk \\
\overline{\overline{q}} &= a + bi + aj - dk
\end{aligned}
$$

Negetive versions are proven later with Lemma 5. Q.E.D.

2.5.6 When $N(q) = 0$

Another thing to consider is when $N(q) = 0$. This is quite simple to find cases for (The surd is left out because it doesn't make a difference if the sum inside is 0):

$$\begin{aligned} N(q) &= 0 \\ ((a+c)^2 + (b-d)^2)((a-c)^2 + (b+d)^2) &= 0 \end{aligned}$$

Lemma 2 (When $N(q)$ is 0). *The cases for when that can equal 0 are:*

$$\begin{gathered} a = 0 \wedge b = 0 \wedge c = 0 \wedge d = 0 \\ a + c = 0 \wedge b - d = 0 \\ a - c = 0 \wedge b + d = 0 \end{gathered}$$

Proof. If $a + c = 0, b - d = 0$

$$\begin{aligned} N(q) &= 0 \\ ((a+c)^2 + (b-d)^2)((a-c)^2 + (b+d)^2) &= 0 \\ (0)((a-c)^2 + (b+d)^2) &= 0 \\ 0 &= 0 \end{aligned}$$

If $a - c = 0, b + d = 0$

$$\begin{aligned} N(q) &= 0 \\ ((a+c)^2 + (b-d)^2)((a-c)^2 + (b+d)^2) &= 0 \\ ((a+c)^2 + (b-d)^2)(0) &= 0 \\ 0 &= 0 \end{aligned}$$

Q.E.D.

Finding out when N can equal 0 is useful because it helps us find 0 divisors.

Chapter 3

Solutions to common problems

3.1 0 divisors ($ax = 0$)

These are really easy to find. Any number that is a 0 divisor has an N of 0. We have a list of all the cases where $N(q) =$ in Lemma 2 and there is a formula for how to generate these 0 divisors.

Lemma 3 (All the 0 divisors' possible forms)**.**

$$\begin{aligned}
\pm(n + mi + nj - mk) \\
\pm(n - mi + nj + mk) \\
\pm(n + mi - nj + mk) \\
\pm(-n + mi + nj + mk)
\end{aligned}$$

Claim. All of the 0 divisor forms in Lemma 3 are 0 divisors

Proof.

$$N(n+mi+nj-mk) = \sqrt{((n+n)^2+(m-(-m))^2)((n-n)^2+(m-m)^2)}$$
$$N(n+mi+nj-mk) = \sqrt{((n+n)^2+(m+m)^2)(0)}$$
$$N(n+mi+nj-mk) = 0$$

$$N(n-mi+nj+mk) = \sqrt{((n+n)^2+(-m-m)^2)((n-n)^2+(-m+m)^2)}$$
$$N(n-mi+nj+mk) = \sqrt{((n+n)^2+(-m-m)^2)(0)}$$
$$N(n-mi+nj+mk) = 0$$

$$N(n+mi-nj+mk) = \sqrt{((n+(-n))^2+(m-m)^2)((n-(-n))^2+(m+m)^2)}$$
$$N(n+mi-nj+mk) = \sqrt{(0)((n+n)^2+(m+m)^2)}$$
$$N(n+mi-nj+mk) = 0$$

$$N(-n+mi+nj+mk) = \sqrt{((-n+n)^2+(m-m)^2)((-n-n)^2+(m+m)^2)}$$
$$N(-n+mi+nj+mk) = \sqrt{(0)((-n-n)^2+(m+m)^2)}$$
$$N(-n+mi+nj+mk) = 0$$

All the negetive versions are proven with Lemma 5. Q.E.D.

Claim. The conjugate formulas listed in Formula 9 do satisfy the quality listed in Lemma 1

Proof.

$$
\begin{aligned}
&(n+mi+nj-mk)(n-mi-nj-mk)\\
=&n^2-nmi-n^2j-nmk\\
&+nmi+m^2+nmk-m^2j\\
&+n^2j+nmk-n^2+nmi\\
&-nmk+m^2j-nmi-m^2\\
=&0
\end{aligned}
$$

$$
\begin{aligned}
&(n-mi+nj+mk)(n+mi-nj+mk)\\
=&n^2+nmi-n^2j+nmk\\
&-nmi+m^2-nmk-m^2j\\
&+n^2j-nmk-n^2-nmi\\
&+nmk+m^2j+nmi-m^2\\
=&0
\end{aligned}
$$

$$
\begin{aligned}
&(n+mi-nj+mk)(n-mi+nj+mk)\\
=&n^2-nmi+n^2j+nmk\\
&+nmi+m^2-nmk+m^2j\\
&-n^2j-nmk-n^2+nmi\\
&+nmk-m^2j-nmi-m^2\\
=&0
\end{aligned}
$$

$$
\begin{aligned}
&(-n+mi+nj+mk)(-n-mi-nj+mk)\\
=&n^2+nmi+n^2j-nmk\\
&-nmi+m^2+nmk+m^2j\\
&-n^2j+nmk-n^2-nmi\\
&-nmk-m^2j+nmi-m^2\\
=&0
\end{aligned}
$$

Q.E.D.

Lemma 4 (*N of a 0 divisor's conjugate*)**.** *If q is a 0 divisor, then $\overline{q}$ must be a 0 divisor too.*

Proof.

$$\begin{aligned} N(q) &= q\overline{q} \text{ assume } q \text{ is a 0 divisor and not 0} \\ 0 &= q\overline{q} \\ q &\neq \overline{q} \neq 0 \\ \therefore\ &\overline{q} \text{ must be a 0 divisor} \end{aligned}$$

Q.E.D.

Lemma 5 (Negetive versions of 0 divisors are also 0 divisors)**.** *If q is a 0 divisor, then so are $-q$ and $-\overline{q}$*

Proof.

$$\begin{aligned} \text{let } q &\text{ be a 0 divisor} \\ \therefore N(q) &= q\overline{q} = 0 \\ q\overline{q} &= 0 \\ -q\overline{q} &= -0 \\ q(-\overline{q}) &= -0 \\ -q(-\overline{q}) &= -(-0) \end{aligned}$$

Q.E.D.

By that proof, all the 0 divisor conjugate formulas can just as well be defined as the negative versions of themselves.

3.2 Inverses (q^{-1})

Formula 10 (Inverse of a commutative quaternion)**.**

$$q^{-1} = \frac{\overline{q}}{N(q)}$$

Proof.

$$N(q) = q\overline{q}$$
$$q = \frac{N(q)}{\overline{q}}$$
$$q^{-1} = \frac{\overline{q}}{N(q)}$$

Q.E.D.

This means that every number for which $N(q) \neq 0$ has an inverse. In other words, every number that isn't a 0 divisor has an inverse and can therefore be divided by. Any 0 divisor can not be divided/cannot have an inverse and from here on out, any attempt to divide by/take the inverse of a 0 divisor will be treated like a division by 0.

Since we can't divide by 0 divisors, we can also just define this as:

$$q^{-1} = \frac{\frac{L(q)}{N(q)}}{N(q)}$$
$$q^{-1} = \frac{L(q)}{N(q)^2}$$

Since there are non-0 numbers that cannot be divided by, this makes this number system not a division algebra.

3.3 Square roots of -1 ($q^2 = -1$)

One of the most common problems for these fancy number systems is finding out which numbers, when squared, yield -1. Let's solve this.

Claim. The only numbers in the commutative quaternions that square to -1 are $\pm i, \pm k$

Proof. Take Formula 1, plug in the desired results and match coefficients:

$$-1 = (a + bi + cj + dk)^2$$
$$-1 + 0i + 0j + 0k = a^2 - b^2 + c^2 - d^2 + (2ab - 2cd)i + (2ac + 2bd)j + (2ad - 2bc)k$$
$$\therefore a^2 - b^2 + c^2 - d^2 = -1,\ 2ab - 2cd = 0,\ 2ac + 2bd = 0,\ 2ad - 2bc = 0$$

Now, we have a system of simultaneous equations and we can solve for the solutions from here. The real solutions are:

$$a = 0 \wedge b = 1 \wedge c = 0 \wedge d = 0$$
$$a = 0 \wedge b = -1 \wedge c = 0 \wedge d = 0$$
$$a = 0 \wedge b = 0 \wedge c = 0 \wedge d = 1$$
$$a = 0 \wedge b = 0 \wedge c = 0 \wedge d = -1$$

Q.E.D.

3.4 Idempotent numbers ($q^2 = q$)

Unlike its regular quaternion counterpart, this number system actually has idempotent elements.

Claim. The commutative quaternions have non-trivial idempotent elements.

Proof. Start by applying Formula 1, plug in in the desired results and match coefficients:

$$a + bi + cj + dk = (a + bi + cj + dk)^2$$
$$a + bi + cj + dk = a^2 - b^2 + c^2 - d^2 + (2ab - 2cd)i + (2ac + 2bd)j + (2ad - 2bc)k$$
$$\therefore a^2 - b^2 + c^2 - d^2 = a,\ 2ab - 2cd = b,\ 2ac + 2bd = c,\ 2ad - 2bc = d$$

We have a system of simultaneous equations and we can solve for a, b, c and d. And the real solutions are as follows:

$$a = 0 \wedge b = 0 \wedge c = 0 \wedge d = 0$$
$$a = 1 \wedge b = 0 \wedge c = 0 \wedge d = 0$$
$$a = \frac{1}{2} \wedge b = 0 \wedge c = \frac{1}{2} \wedge d = 0$$
$$a = \frac{1}{2} \wedge b = 0 \wedge c = -\frac{1}{2} \wedge d = 0$$

Q.E.D.

Which we can see is just the trivial ones (1 and 0) and the exact same idempotent numbers as the split complex numbers: $\frac{1 \pm j}{2}$. This means that we actually don't have enough numbers to construct a diagonal basis for the commutative quaternions.

3.5 Nilpotent numbers ($q^2 = 0$)

Claim. The commutative quaternions have no non-trivial nilpotent elements.

Proof. Start by applying Formula 1, plug in in the desired results and match coefficients:

$$0 = (a + bi + cj + dk)^2$$
$$0 + 0i + 0j + 0k = a^2 - b^2 + c^2 - d^2 + (2ab - 2cd)i + (2ac + 2bd)j + (2ad - 2bc)k$$
$$\therefore a^2 - b^2 + c^2 - d^2 = 0,\ 2ab - 2cd = 0,\ 2ac + 2bd = 0,\ 2ad - 2bc = 0$$

We have a system of simultaneous equations. The real solutions are:

$$a = 0 \wedge b = 0 \wedge c = 0 \wedge d = 0$$

Only 0, which is trivial. Therefore, there are no non-trivial nilpotent elements.

Q.E.D.

3.6 Exponential (e^q)

Also known as $\exp(q)$, Euler proved this for the complex numbers and this has become a staple in all sorts of complex numbers. Let's start with e^{ix}.

Claim. $e^{xi} = \cos(x) + \sin(x)i$

Proof. Follow e^{ix} by calculating it's Taylor series and simplify.

$$\begin{aligned}
e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} \\
&= \frac{1}{0!} + \frac{ix}{1!} + \frac{-x^2}{2!} + \frac{-x^3 i}{3!} + \frac{x^4}{4!} + \frac{x^5 i}{5!} + \dots \\
&= \frac{1}{0!} + \frac{x}{1!}i - \frac{x^2}{2!} - \frac{x^3}{3!}i + \frac{x^4}{4!} + \frac{x^5}{5!}i + \dots \\
&= (\frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots) + (\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots)i \\
&= \cos(x) + \sin(x)i
\end{aligned}$$

Q.E.D.

As we can see, exactly as Euler proved. The same method can be applied to k as k shares the same properties as i when it is squared.

Next up is j.

Claim. $e^{xj} = \cosh(x) + \sinh(x)j$

Proof. Follow e^{jx} by calculating it's Taylor series and simplify.

$$
\begin{aligned}
e^{jx} &= \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} \\
&= \frac{1}{0!} + \frac{jx}{1!} + \frac{x^2}{2!} + \frac{x^3 j}{3!} + \frac{x^4}{4!} + \frac{x^5 j}{5!} + \dots \\
&= \frac{1}{0!} + \frac{x}{1!}j + \frac{x^2}{2!} + \frac{x^3}{3!}j + \frac{x^4}{4!} + \frac{x^5}{5!}j + \dots \\
&= (\frac{1}{0!} + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots) + (\frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots)j \\
&= \cosh(x) + \sinh(x)j
\end{aligned}
$$

Q.E.D.

So, new we have all out components; Let's put them together.

Claim.

$$
\begin{aligned}
&e^{a+bi+cj+dk} \\
&= \left(\frac{1}{2}(e^{a+c}\cos(b-d) + e^{a-c}\cos(b+d))\right) \\
&+ \left(\frac{1}{2}(e^{a+c}\sin(b-d) + e^{a-c}\sin(b+d))\right) i \\
&- \left(\frac{1}{2}(e^{a-c}\cos(b+d) + e^{a+c}\cos(b-d))\right) j \\
&- \left(\frac{1}{2}(e^{a-c}\sin(b+d) + e^{a+c}\sin(b-d))\right) k
\end{aligned}
$$

Proof.

$$
\begin{aligned}
q :=& a + bi + cj + dk \\
e^q =& e^{a+bi+cj+dk} \\
=& e^a \cdot e^{bi} \cdot e^{cj} \cdot e^{dk} \\
=& e^a \cdot (\cos(b) + \sin(b)i) \cdot (\cosh(c) + \sinh(c)j) \cdot (\cos(d) + \sin(d)k) \\
=& e^a \cos(b)\cos(d)\cosh(c) + ke^a \cos(b)\sin(d)\cosh(c) \\
&+ je^a \cos(b)\cos(d)\sinh(c) + jke^a \cos(b)\sin(d)\sinh(c) \\
&+ ie^a \sin(b)\cos(d)\cosh(c) + ike^a \sin(b)\sin(d)\cosh(c) \\
&+ ije^a \sin(b)\cos(d)\sinh(c) + ijke^a \sin(b)\sin(d)\sinh(c) \\
=& e^a \cos(b)\cos(d)\cosh(c) + e^a \cos(b)\sin(d)\cosh(c)k \\
&+ e^a \cos(b)\cos(d)\sinh(c)j - e^a \cos(b)\sin(d)\sinh(c)i \\
&+ e^a \sin(b)\cos(d)\cosh(c)i + e^a \sin(b)\sin(d)\cosh(c)j \\
&- e^a \sin(b)\cos(d)\sinh(c)k + e^a \sin(b)\sin(d)\sinh(c) \\
=& e^a(\cos(b)\cos(d)\cosh(c) + \sin(b)\sin(d)\sinh(c)) \\
&+ e^a(\sin(b)\cos(d)\cosh(c) - \cos(b)\sin(d)\sinh(c))i \\
&+ e^a(\cos(b)\cos(d)\sinh(c) + \sin(b)\sin(d)\cosh(c))j \\
&+ e^a(\cos(b)\sin(d)\cosh(c) - \sin(b)\cos(d)\sinh(c))k \\
=& \left(\frac{1}{2}(e^{a+c}\cos(b-d) + e^{a-c}\cos(b+d))\right) \\
&+ \left(\frac{1}{2}(e^{a+c}\sin(b-d) + e^{a-c}\sin(b+d))\right) i \\
&- \left(\frac{1}{2}(e^{a-c}\cos(b+d) + e^{a+c}\cos(b-d))\right) j \\
&- \left(\frac{1}{2}(e^{a-c}\sin(b+d) + e^{a+c}\sin(b-d))\right) k
\end{aligned}
$$

Q.E.D.

Chapter 4

Exponential representation

4.1 Background

Most of these complex number systems have some sort of polar or otherwise planar form. These are normally written in 1 multiplication with exponents. This system also has one although, it is complicated to get to. Nevertheless, it is worth having it. Many problems such as radicals and logarithms become very easy to solve when you have this form. Note: This form cannot represent 0 divisors but this is to be expected.

4.2 Prerequisites to the proof

4.2.1 Distances of commutative quaternions with 1 imaginary part

Distance of an i commutative quaternion

Claim. $||a + bi|| = \sqrt{a^2 + b^2}$

Proof.

$$\begin{aligned}
&||a + bi|| \\
=&\sqrt{N(a + bi)} \\
=&\sqrt{\sqrt{((a + 0)^2 + (b - 0)^2)((a - 0)^2 + (b + 0)^2)}} \\
=&\sqrt{\sqrt{(a^2 + b^2)^2}} \\
=&\sqrt{a^2 + b^2}
\end{aligned}$$

Q.E.D.

Same as the complex numbers.

Distance of an j commutative quaternion

Claim. $||a + bj|| = \sqrt{a^2 - b^2}$

Proof.

$$
\begin{aligned}
&||a + bj|| \\
=&\sqrt{N(a + bj)} \\
=&\sqrt{\sqrt{((a + b)^2 + (0 - 0)^2)((a - b)^2 + (0 + 0)^2)}} \\
=&\sqrt{\sqrt{(a + b)(a - b)(a + b)(a - b)}} \\
=&\sqrt{\sqrt{(a^2 - b^2)^2}} \\
=&\sqrt{a^2 - b^2}
\end{aligned}
$$

Q.E.D.

Same as the split complex numbers.

4.2.2 Exponentiation and planar forms

Polar form of i commutative quaternion

Claim. Any i commutative quaternion can be represented as:

$$
\begin{aligned}
&a + bi \\
=&\sqrt{a^2 + b^2}e^{\text{atan2}(b,a)}
\end{aligned}
$$

Proof. Since an i commutative quaternion is isomorphic to the plane of complex numbers, it's polar form is the same.

It can simply be represented as point on the unit circle multiplied by the distance of the number away from the origin. This can be represented as:

$$
\begin{aligned}
&a + bi \\
=&\sqrt{a^2 + b^2}e^{\text{atan2}(b,a)}
\end{aligned}
$$

Q.E.D.

Which is the same as the complex numbers.

Planar form of j commutative quaternion

Claim. Any j commutative quaternion that isn't a 0 divisor can be represented as:

$$
\begin{aligned}
&a + bj \\
=&\sqrt{a^2 - b^2} e^{\frac{1}{2}\ln(1+\frac{b}{a})j - \frac{1}{2}\ln(1-\frac{b}{a})j}
\end{aligned}
$$

Proof. Since a j commutative quaternion is isomorphic to the plane of split complex numbers, it's planar form is the same.

Take a j commutative quaternion of the form $a + bj$. This can be represented as some distance away from the origin and a hyperbolic angle of rotation. This can be represented as:

$$
\begin{aligned}
&a + bj \\
=&\sqrt{a^2 - b^2} e^{\tanh^{-1}(\frac{b}{a})j}
\end{aligned}
$$

That $\tanh^{-1}(\frac{b}{a})$ is better represented in it's decomposed form: $\frac{1}{2}\ln(1 + \frac{b}{a}) - \frac{1}{2}\ln(1 - \frac{b}{a})$

Therefore, any j commutative quaternion that isn't a 0 divisor (Or split complex number that isn't a 0 divisor) can be represented as:

$$
\begin{aligned}
&a + bj \\
=&\sqrt{a^2 - b^2} e^{\frac{1}{2}\ln(1+\frac{b}{a})j - \frac{1}{2}\ln(1-\frac{b}{a})j}
\end{aligned}
$$

Q.E.D.

Which is the same as the split complex numbers.

4.3 The form itself

This form is very long but it works.

Formula 11. *Any commutative quaternion can be represented as:*

$$\begin{aligned}&a+bi+cj+dk\\=&||a+bi+cj+dk||\\&\cdot e^{\frac{1}{2}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i}\\&\cdot e^{\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j}\\&\cdot e^{\frac{1}{2}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k}\end{aligned}$$

As long as that commutative quaternion is not a 0 divisor.

Proof. Start with taking any commutative quaternion and factoring out a j from the last 2 terms, representing it as something like 2 complex numbers with the last one multiplied by the split complex number j (In some sort of strange Cayley Dickson form):

$$\begin{aligned}&a+bi+cj+dk\\=&(a+bi)+(c-di)j\end{aligned}$$

Next, notice that that is basically a j commutative quaternion but instead of real coefficients, it has "complex" ones. Therefore, it can be decomposed into the planar form (Except for 0 divisors):

$$\begin{aligned}=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\tanh^{-1}(\frac{c-di}{a+bi}))j}\\=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln\left(1+\frac{c-di}{a+bi}\right)-\frac{1}{2}\ln\left(1-\frac{c-di}{a+bi}\right))j}\end{aligned}$$

Next, add the 1s together with the complex fractions, apply the laws of logarithms, expand out and simplify:

$$\begin{aligned}=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln\left(\frac{a+bi}{a+bi}+\frac{c-di}{a+bi}\right)-\frac{1}{2}\ln\left(\frac{a+bi}{a+bi}-\frac{c-di}{a+bi}\right))j}\\=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln\left(\frac{a+bi+c-di}{a+bi}\right)-\frac{1}{2}\ln\left(\frac{a+bi-c+di}{a+bi}\right))j}\\=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}(\ln(a+bi+c-di)-\ln(a+bi))-\frac{1}{2}(\ln(a+bi-c+di)-\ln(a+bi)))j}\\=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln(a+bi+c-di)-\frac{1}{2}\ln(a+bi)-\frac{1}{2}\ln(a+bi-c+di)+\frac{1}{2}\ln(a+bi))j}\\=&\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln(a+bi+c-di)-\frac{1}{2}\ln(a+bi-c+di))j}\end{aligned}$$

Next, collect coefficients and calculate the i commutative quaternion ("complex") logarithms. This is done by converting the i commutative quaternions to their polar forms and applying laws of logarithms from there:

$$=\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln((a+c)+(b-d)i)-\frac{1}{2}\ln((a-c)+(b+d)i))j}$$
$$=\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}\ln(\sqrt{(a+c)^2+(b-d)^2}e^{\operatorname{atan2}(b-d,a+c)i})-\frac{1}{2}\ln(\sqrt{(a-c)^2+(b+d)^2}e^{\operatorname{atan2}(b+d,a-c)i}))j}$$
$$=\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}(\ln(\sqrt{(a+c)^2+(b-d)^2})+\ln(e^{\operatorname{atan2}(b-d,a+c)i}))-\frac{1}{2}(\ln(\sqrt{(a-c)^2+(b+d)^2})+\ln(e^{\operatorname{atan2}(b+d,a-c)i})))j}$$
$$=\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$

Now, we're done with that exponential part for now. To deal with the distance part. Start by expanding the expression, simplifying and collecting coefficients:

$$=\sqrt{(a+bi)^2-(c-di)^2}e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$
$$=\sqrt{(a^2+2abi-b^2)-(c^2-2cdi-d^2)}$$
$$\cdot e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$
$$=\sqrt{a^2+2abi-b^2-c^2+2cdi+d^2}$$
$$\cdot e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$
$$=\sqrt{(a^2-b^2-c^2+d^2)+(2ab+2cd)i}$$
$$\cdot e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$

Next, take the "complex" number in the radical, convert it to it's polar form and simplify:

$$=\sqrt{\sqrt{(a^2-b^2-c^2+d^2)^2+(2ab+2cd)^2}e^{\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i}}$$
$$\cdot e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$
$$=\sqrt[4]{(a^2-b^2-c^2+d^2)^2+(2ab+2cd)^2}e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i}$$
$$\cdot e^{(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$
$$=\sqrt[4]{(a^2-b^2-c^2+d^2)^2+(2ab+2cd)^2}$$
$$\cdot e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i+(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$

Next, notice that that radical is just a different way to write $N(a+bi+cj+dk)^2$.

Thus, it is the distance function:

$$=\sqrt[4]{N(q)^2} \cdot e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i+(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$

$$=||a+bi+cj+dk|| \cdot e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i+(\frac{1}{2}(\frac{1}{2}\ln((a+c)^2+(b-d)^2)+\operatorname{atan2}(b-d,a+c)i)-\frac{1}{2}(\frac{1}{2}\ln((a-c)^2+(b+d)^2)+\operatorname{atan2}(b+d,a-c)i))j}$$

Next, to simplify the expression. Multiply in the $\frac{1}{2}$s and multiply in the j at the end:

$$=||a+bi+cj+dk|| \cdot e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i+((\frac{1}{4}\ln((a+c)^2+(b-d)^2)+\frac{1}{2}\operatorname{atan2}(b-d,a+c)i)-(\frac{1}{4}\ln((a-c)^2+(b+d)^2)+\frac{1}{2}\operatorname{atan2}(b+d,a-c)i))j}$$

$$=||a+bi+cj+dk|| \cdot e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i+\frac{1}{4}\ln((a+c)^2+(b-d)^2)j+\frac{1}{2}\operatorname{atan2}(b-d,a+c)ij-\frac{1}{4}\ln((a-c)^2+(b+d)^2)j-\frac{1}{2}\operatorname{atan2}(b+d,a-c)ij}$$

$$=||a+bi+cj+dk|| \cdot e^{\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2)i+\frac{1}{4}\ln((a+c)^2+(b-d)^2)j-\frac{1}{2}\operatorname{atan2}(b-d,a+c)k-\frac{1}{4}\ln((a-c)^2+(b+d)^2)j+\frac{1}{2}\operatorname{atan2}(b+d,a-c)k}$$

Finally, group up the matching coefficients and factor out some halves:

$$=||a+bi+cj+dk|| \cdot e^{(\frac{1}{2}\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i+(\frac{1}{4}\ln((a+c)^2+(b-d)^2)-\frac{1}{4}\ln((a-c)^2+(b+d)^2))j+(\frac{1}{2}\operatorname{atan2}(b+d,a-c)-\frac{1}{2}\operatorname{atan2}(b-d,a+c))k}$$

$$=||a+bi+cj+dk|| \cdot e^{\frac{1}{2}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i+\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j+\frac{1}{2}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k}$$

And that's it. You can finally write it neatly as:

$$\begin{aligned} & a+bi+cj+dk \\ =& ||a+bi+cj+dk|| \\ & \cdot e^{\frac{1}{2}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i} \\ & \cdot e^{\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\ & \cdot e^{\frac{1}{2}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k} \end{aligned}$$

Q.E.D.

This has a drawback that is also in the split complex numbers: it cannot represent any 0 divisors. At all. This is to be expected though.

A thing to remember is that the distance function can return imaginary results (As the square root might be negative). Thus, to further calculate the planar form, it may be worth further converting that negative number into it's polar form (With imaginary part i or k) and working from there.

For example, the number -1 *should* have a distance of -1 away from the origin (Because $\sqrt[4]{1}$ does have -1 as an answer and because it lies on the conjugate mesh of the unit commutative quaternion and not the main one). Which makes this form hold together. Without having negative or imaginary distances, this function would not work.

4.4 Problems that can be now solved using this form

4.4.1 Square roots ($\sqrt{q}$)

The square root of any arbitrary commutative quaternion is now trivial to solve for by applying Formula 11.

Claim.

$$
\begin{aligned}
&\sqrt{a+bi+cj+dk} \\
=&\sqrt{||a+bi+cj+dk||} \\
&\cdot e^{\frac{1}{4}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i} \\
&\cdot e^{\frac{1}{8}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\
&\cdot e^{\frac{1}{4}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\sqrt{a+bi+cj+dk} \\
=&\sqrt{||a+bi+cj+dk||} \\
&\cdot \sqrt{e^{\frac{1}{2}(\text{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i}} \\
&\cdot \sqrt{e^{\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j}} \\
&\cdot \sqrt{e^{\frac{1}{2}(\text{atan2}(b+d,a-c)-\text{atan2}(b-d,a+c))k}} \\
=&\sqrt{||a+bi+cj+dk||} \\
&\cdot e^{\frac{1}{4}(\text{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i} \\
&\cdot e^{\frac{1}{8}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\
&\cdot e^{\frac{1}{4}(\text{atan2}(b+d,a-c)-\text{atan2}(b-d,a+c))k}
\end{aligned}
$$

Q.E.D.

4.4.2 nth roots ($\sqrt[n]{q}$)

In fact, now that we have an exponential form, all radicals become trivial using Formula 11.

Claim.

$$
\begin{aligned}
&\sqrt[n]{a+bi+cj+dk} \\
=&\sqrt[n]{||a+bi+cj+dk||} \\
&\cdot e^{\frac{1}{2n}(\text{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i} \\
&\cdot e^{\frac{1}{4n}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\
&\cdot e^{\frac{1}{2n}(\text{atan2}(b+d,a-c)-\text{atan2}(b-d,a+c))k}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\sqrt[n]{a+bi+cj+dk} \\
=&\sqrt[n]{||a+bi+cj+dk||} \\
&\cdot \sqrt[n]{e^{\frac{1}{2}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i}} \\
&\cdot \sqrt[n]{e^{\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j}} \\
&\cdot \sqrt[n]{e^{\frac{1}{2}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k}} \\
=&\sqrt[n]{||a+bi+cj+dk||} \\
&\cdot e^{\frac{1}{2n}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i} \\
&\cdot e^{\frac{1}{4n}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\
&\cdot e^{\frac{1}{2n}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k}
\end{aligned}
$$

Q.E.D.

4.4.3 Arbitrary exponents ($q^n,\ n \in \mathbb{R}$)

This too, becomes trivial using Formula 11.

Claim.

$$
\begin{aligned}
&a+bi+cj+dk^n \\
=&||a+bi+cj+dk||^n \\
&\cdot e^{\frac{n}{2}(\operatorname{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i} \\
&\cdot e^{\frac{n}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\
&\cdot e^{\frac{n}{2}(\operatorname{atan2}(b+d,a-c)-\operatorname{atan2}(b-d,a+c))k}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
& a+bi+cj+dk^n \\
=& ||a+bi+cj+dk||^n \\
& \cdot e^{\frac{1}{2}(\operatorname{atan2}(2ab+2cd, a^2-b^2-c^2+d^2))i^n} \\
& \cdot e^{\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j^n} \\
& \cdot e^{\frac{1}{2}(\operatorname{atan2}(b+d, a-c)-\operatorname{atan2}(b-d, a+c))k^n} \\
=& ||a+bi+cj+dk||^n \\
& \cdot e^{\frac{n}{2}(\operatorname{atan2}(2ab+2cd, a^2-b^2-c^2+d^2))i} \\
& \cdot e^{\frac{n}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j} \\
& \cdot e^{\frac{n}{2}(\operatorname{atan2}(b+d, a-c)-\operatorname{atan2}(b-d, a+c))k}
\end{aligned}
$$

Q.E.D.

4.4.4 Natural logarithm ($\ln(q)$)

Since the planar form is all in base e, it is easy to calculate.
Claim.

$$
\begin{aligned}
& \ln(a+bi+cj+dk) \\
=& \ln(||a+bi+cj+dk||) \\
& +\frac{1}{2}(\operatorname{atan2}(2ab+2cd, a^2-b^2-c^2+d^2))i \\
& +\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j \\
& +\frac{1}{2}(\operatorname{atan2}(b+d, a-c)-\operatorname{atan2}(b-d, a+c))k
\end{aligned}
$$

Proof.

$$
\begin{aligned}
&\ln(a + bi + cj + dk)\\
=&\ln(||a + bi + cj + dk||)\\
&+\ln(e^{\frac{1}{2}(\text{atan2}(2ab+2cd,a^2-b^2-c^2+d^2))i})\\
&+\ln(e^{\frac{1}{4}(\ln((a+c)^2+(b-d)^2)-\ln((a-c)^2+(b+d)^2))j})\\
&+\ln(e^{\frac{1}{2}(\text{atan2}(b+d,a-c)-\text{atan2}(b-d,a+c))k})\\
=&\ln(||a + bi + cj + dk||)\\
&+\frac{1}{2}(\text{atan2}(2ab + 2cd, a^2 - b^2 - c^2 + d^2))i\\
&+\frac{1}{4}(\ln((a + c)^2 + (b - d)^2) - \ln((a - c)^2 + (b + d)^2))j\\
&+\frac{1}{2}(\text{atan2}(b + d, a - c) - \text{atan2}(b - d, a + c))k
\end{aligned}
$$

Q.E.D.

4.4.5 Arbitrary logarithms ($\log_n(q)$)

Since we have a definition for the natural logarithm, by rules of logarithms, we can say that:

$$\log_a(b) = \frac{\log_n(b)}{\log_n(a)}$$

And since we have some n based log (Which is the natural logarithm or log bases e), we can say that:

$$
\begin{aligned}
&\log_n(q)\\
=&\frac{\ln(q)}{\ln(n)}
\end{aligned}
$$

And since this law holds, for all n, you can take arbitrary logarithms with arbitrary bases too.

This will not be formally proven since it is trivial.

Chapter 5

Isomorphisms between these numbers and various groups

This number system has a few isomorphisms between different groups and this will list them and what elements are isomorphic to it.

I'm sure there are more than what I listed here but this is what I found:

Klein 4 group

The set $\{1, -1, j, -j\}$ is isomorphic to the Klein 4 group. Which itself is isomorphic to $\mathbb{R}[X]/\langle X^2 - 1\rangle$ and to the plane of split complex numbers.

Isomorphism to the plane of complex numbers

The sets $\{1, -1, i, -i\}$ and $\{1, -1, k, -k\}$ are both isomorphic to the group $\mathbb{R}[X]/\langle X^2 + 1\rangle$ or to the plane of complex numbers.

Chapter 6

Final remarks

It has occurred to me that after writing this paper, these numbers are quite similar to the Tessarines (Or bicomplex numbers). Which makes a bit of the work I do here a little useless. But, I did a lot of stuff (I.E: Exponential form) that is not proven or mentioned in any piece of literature that I read about the Tessarines. And, the commutative quaternions act slightly differently from the Tessarines so I still consider them unique and not a waste of time but that might just be the sunk cost fallacy talking.

These numbers are very interesting to me. I learnt a lot about mathematics in the process of writing this paper. I believe that almost every problem presented by these numbers has been solved.

I am pretty sure these numbers could become even more interesting if you had the imaginary units be units over some set other than the real numbers. For instance, making it a set over the dual numbers could introduce interesting non-trivial nilpotents and other interesting properties that I haven't looked at.

These numbers are not practical, useful or any such thing. They are more a way to show that mathematics can be fun. No matter how useless or silly the concept may be.

Appendix A

Table of multiplication

This is a simple table outlying the rules of multiplication of the commutative quaternions.

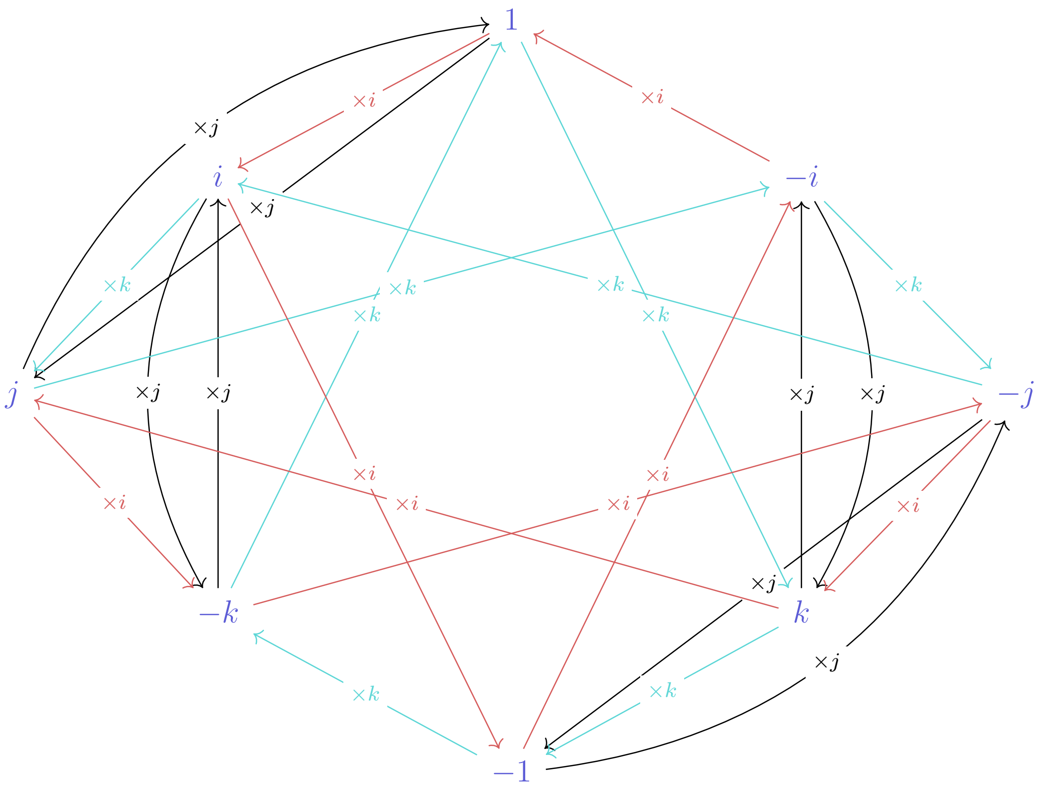
$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	$-k$	j
j	j	$-k$	1	$-i$
k	k	j	$-i$	-1

Appendix B

Diagram of multiplication

This is essentially a Cayley graph of the multiplication rules of the commutative quaternions.

Purple represents the values being multiplied, black represents multiplying by j, red means multiplying by i and blue means multiplying by k.



Glossary

0 commutative quaternion A commutative quaternion where all of it's parts of 0. Equivilant to the real number 0. It is a scalar commutative quaternion.

i commutative quaternion A commutative quaternion that takes the form of $a + bi + 0j + 0k$. Isomorphic to the plane of complex numbers..

j commutative quaternion A commutative quaternion that takes the form of $a + 0i + bj + 0k$. Isomorphic to the plane of split complex numbers..

k commutative quaternion A commutative quaternion that takes the form of $a + 0i + 0j + bk$. Isomorphic to the plane of complex numbers..

Scalar quaternion A commutative quaternion made entirely only out of real parts (A real number only).

Vector quaternion A commutative quaternion made entirely only out of imaginary parts.

Bibliography

[1] https://en.wikipedia.org/wiki/Split-quaternion

[2] https://en.wikipedia.org/wiki/Split-complex_number

[3] https://en.wikipedia.org/wiki/Quaternion

[4] https://en.wikipedia.org/wiki/Inverse_hyperbolic_functions

[5] https://en.wikipedia.org/wiki/polar_decomposition

[6] https://en.wikipedia.org/wiki/Klein_four-group

[7] https://en.wikipedia.org/wiki/Complex_number

[8] https://en.wikipedia.org/wiki/Zero_divisor

[9] https://math.stackexchange.com/questions/1030737/exponential-function-of-quaternion-derivation

[10] https://www.youtube.com/watch?v=dvI7dGXFgm8